

LATTICE-VALUED BOREL MEASURES. II

BY

SURJIT SINGH KHURANA

ABSTRACT. Let T be a completely regular Hausdorff space, $C_b(T)$ the set of all bounded real-valued continuous functions on T , E a boundedly monotone complete ordered vector space, and $\varphi: C_b(T) \rightarrow E$ a positive linear map. It is proved that under certain conditions there exist σ -additive, τ -smooth or tight E -valued measures on T which represent φ .

Let T be a completely regular Hausdorff space, $C_b(T)$ the vector-lattice of all bounded, real-valued functions on T . Let E be a boundedly complete partially ordered vector space and $\varphi: C_b(T) \rightarrow E$ a positive linear map, i.e., $f \in C_b(T)$, $f \geq 0$ implies $\varphi(f) \geq 0$. In case T is compact, it is known ([2], [5]) that a quasi-regular Borel measure μ on T which represents φ , i.e., $\varphi(f) = \int f d\mu$, $\forall f \in C(T)$, all continuous real-valued functions on T (see [4], [5] for details). The more general case of a Hausdorff completely regular space is the aim of study in this paper.

For a topological space Y let $\mathfrak{B}(Y)$ be the σ -algebra of all Borel subsets of Y and $\mathfrak{B}_0(Y)$ the σ -algebra of all Baire subsets of Y (that is, the smallest σ -algebra which makes each bounded continuous function on Y measurable). Let $B(Y)$ ($B_0(Y)$) be the space of all bounded Borel (Baire) measurable functions on Y . For basic facts about vector lattices we refer to [1] (see also [2], [4]–[9]). We shall make use of the result proved in [2], that if a boundedly σ -complete vector lattice E contains a vector subspace F which is monotone order σ -closed and if F contains a vector sublattice G of E , then F contains the order σ -closure of G . If S is a Stonian (σ -Stonian) compact Hausdorff space we define a mapping $\psi: B(S) \rightarrow C(S)$ ($\psi_1: B_0(S) \rightarrow C(S)$), $\psi(f) = f$ ($\psi_1(f) = f$) except on a meagre subset of S . It is easy to verify that ψ and ψ_1 are positive order σ -continuous linear maps and for any increasing net $\{f_\alpha\} \subset C(S)$, with $\sup f_\alpha = f \in B(S)$, $\psi(f) = \sup \psi(f_\alpha)$, $\sup$ being taken in the boundedly complete vector lattice $C(S)$ [2] (these maps are called Loomis-Sikorski maps [5]).

Throughout the paper any E -valued measure μ on any σ -algebra is required

Received by the editors July 7, 1975 and, in revised form, April 28, 1976.

AMS (MOS) subject classifications (1970). Primary 28A55, 46G10, 28A45; Secondary 28A25, 28A30.

Key words and phrases. τ -smooth measures, tight measures, weakly σ -distributive lattices, monotone order σ -continuous, order σ -continuous, monotone order σ -closed, order σ -closed.

© American Mathematical Society 1978

to be nonnegative and σ -additive with respect to the order of E , i.e., whenever $\{F_n\}$, $1 \leq n < \infty$, is a monotonic sequence in the σ -algebra then $\mu(\bigcup_{n=1}^{\infty} F_n) = \bigvee_{n=1}^{\infty} \mu(F_n)$. Integration with respect to these measures is taken in the sense of [4], [5]. Throughout this paper $\varphi: C_b(T) \rightarrow E$ is assumed to be a given positive linear map, T being a Hausdorff completely regular space, and E a boundedly monotone complete vector lattice assumed to be over the field of real numbers. Denoting by X the Stone-Čech compactification of T , we get a positive linear map $\bar{\varphi}: C(X) \rightarrow E$, $\bar{\varphi}(f) = \varphi(f|_T)$, which is represented by a quasi-regular Borel measure $\bar{\mu}$ on X , by the

PROPOSITION 1 (WRIGHT [4], [5]). *Given a positive linear map $\varphi: C_b(T) \rightarrow E$, there exists a unique quasi-regular E -valued Borel measure $\bar{\mu}$ on X such that*

$$\varphi(f|_X) = \int f d\bar{\mu}, \quad \text{for all } f \in C(X).$$

PROOF. This theorem is proved in [5]. By quasi-regularity we mean that for any open subset V of X , $\bar{\mu}(V) = \sup\{\bar{\mu}(C): C \text{ compact, } C \subset V\}$.

To get a measure on T we first note the following result.

LEMMA 2. (i) $\mathfrak{B}_0(T) = \{E \cap T: E \in \mathfrak{B}_0(X)\}$.

(ii) $\mathfrak{B}(T) = \{E \cap T: E \in \mathfrak{B}(X)\}$.

Proof is similar to Lemma C [6].

COROLLARY 3. *There is a unique E -valued Baire measure μ on X which represents φ , i.e.,*

$$\varphi(f) = \int f d\mu, \quad \text{for all } f \in C_b(T),$$

if and only if $\bar{\mu}(A) = 0$ for any $A \in \mathfrak{B}_0(X)$ with $A \cap T = \emptyset$.

PROOF. If $\bar{\mu}(A) = 0, \forall A \in \mathfrak{B}_0(X)$, with $A \cap T = \emptyset$, then defining $\mu(P) = \bar{\mu}(P_0)$, where $P_0 \cap T = P$ for a $P_0 \in \mathfrak{B}_0(X)$, for any $P \in \mathfrak{B}_0(T)$, it is easy to see that μ is well defined, is countably additive, and $\int f d\bar{\mu} = \int f|_X d\mu, \forall f \in C(X)$. Conversely, if there is such a μ , then $\int f d\bar{\mu} = \int f|_X d\mu, \forall f \in C(X)$. This means $\{f \in B_0(X): \int f d\bar{\mu} = \int f|_X d\mu\}$ contains $C(X)$ and is monotone order σ -closed. Thus $\int f d\bar{\mu} = \int f|_X d\mu, \forall f \in \mathfrak{B}_0(X)$, and so the result follows (cf. [6, Theorem E]).

For each positive e in E , let

$$E[e] = \{a \in E: \exists \lambda > 0 \text{ such that } -\lambda e \leq a \leq \lambda e\}.$$

Thus $E[e]$ is an order-unit space and can be equipped with the order-unit norm. Since φ and $\bar{\mu}$ take their values in E ($\varphi(1)$) there is no loss of generality in supposing E is an order-unit space with order unit $e = \varphi(1)$. Let E be equipped with order-unit norm.

PROPOSITION 4. *A sufficient condition for the existence of a unique E -valued Baire measure μ on T which represents φ is that there exists a weakly σ -distributive [6], boundedly σ -complete vector lattice W such that E can be embedded (without alternation of suprema) in W and, whenever $\{f_n\}$ is a monotonic decreasing sequence in $C_b(T)$ with pointwise infimum 0, then $\bigwedge_{n=1}^{\infty} \varphi(f_n) = 0$.*

PROOF. In this case, $\bar{\mu}: B_0(X) \rightarrow E$ is regular [6]. The given condition gives $\bar{\mu}(Z) = 0$ for any zero set Z of X , $Z \cap T = \emptyset$ (by zero-set we mean $f^{-1}\{0\}$, for some $f \in C(X)$). By regularity $\mu(P) = 0$ for any $P \in \mathfrak{B}_0(X)$, $P \cap T = \emptyset$. Corollary 3 now gives the result.

PROPOSITION 5. *A sufficient condition for the existence of a unique E -valued Baire measure μ on T which represents φ is that whenever $\{f_n\}$ is a monotone decreasing sequence in $C_b(T)$ which pointwise converges to 0 then $\|\varphi(f_n)\| \rightarrow 0$.*

PROOF. Proceeding as in [2] we see that $\bar{\varphi}(C(X))$ is embedded, as an ordered vector space, in $C(S)$ for some Stonian compact Hausdorff space S , preserving arbitrary suprema and infima and $\bar{\varphi}(1)$ being the constant function 1 in $C(S)$ (this can also be done by taking MacNeille-Dedekind completion of E [4], [9]). This gives us a positive linear map $\varphi: C_b(T) \rightarrow C(S) \subset B_0(S)$, with pointwise order in $B_0(S)$. Since $B_0(S)$ is trivially weakly σ -distributive, it follows from [8, Theorem 3.4] that φ extends to a linear, positive, monotone order σ -continuous map $\varphi: B_0(T) \rightarrow B_0(S)$. Since $(\psi_1 \circ \varphi)^{-1}(E)$ is monotone order σ -closed and contains the lattice $C_b(T)$, we get $(\psi_1 \circ \varphi)^{-1}(E) = B_0(T)$. Defining $\mu = \psi_1 \circ \varphi|_{\mathfrak{B}_0(T)}$ we get the desired Baire measure. The uniqueness is easy to verify.

REMARK. It is enough to assume in Proposition 4 and 5 that E is boundedly monotone order σ -complete.

DEFINITION. (a) An E -valued measure $\mu: \mathfrak{B}(T) \rightarrow E$ is said to be τ -smooth if, whenever $\{U_\alpha\}$ is an increasing net of open sets, $\mu(\bigcup U_\alpha) = \bigvee_\alpha \mu(U_\alpha)$.

(b) An E -valued measure $\mu: \mathfrak{B}(T) \rightarrow E$ is said to be tight if for any open set U , $\mu(U) = \bigvee \{\mu(C): C \text{ compact}, C \subset U\}$.

We list some properties of these measures.

PROPOSITION 6. (i) *If μ is τ -smooth then for any decreasing net $\{g_\alpha\}$ of bounded, upper semicontinuous, real-valued functions with pointwise $\inf g_\alpha = g \in B(X_0)$, $\int g d\mu = \bigwedge_\alpha \int g_\alpha d\mu$; also for an open set V , in T , $\mu(V) = \bigvee \{\mu(P): P \subset V, P \text{ closed in } T\}$.*

(ii) *If μ is tight, then the following statements hold:*

(a) μ is τ -smooth.

(b) *If a net $\{f_\alpha\}$, in $B(T)$, converges to $f \in B(T)$, uniformly on compact*

subsets of T , $\|f_\alpha\| \leq 1$, $\forall \alpha$, and E is a boundedly complete vector lattice, then $\int f_\alpha d\mu \rightarrow 0$ (order convergence).

(c) If V_1 and V_2 are open subsets of T , then $\mu(V_1 \setminus V_2) = \bigvee \{ \mu(C) : C \text{ compact, } C \subset V_1 \setminus V_2 \}$.

(d) For a bounded, nonnegative, lower semicontinuous function f on T , $\int f d\mu = \bigvee \{ \int g d\mu : 0 \leq g \leq f, g \text{ simple and a combination of characteristic functions of disjoint compact subsets of } T \}$.

PROOF. (i) This follows by using the inequality

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \mu \left(\left\{ x \in T : f(x) \geq \frac{i}{n} \right\} \right) \\ \leq \int f d\mu \leq \frac{1}{n} \mu(T) + \frac{1}{n} \sum_{i=1}^n \mu \left(\left\{ x \in T : f(x) > \frac{i}{n} \right\} \right), \end{aligned}$$

valid for any $n \geq 1$ and for any measurable f , $0 < f < 1$. The regularity property follows from the fact that every point of T has a nbd. base consisting of closed sets.

(ii) (a) is trivially obvious. To prove (b), we have, for any compact subset C of T ,

$$\begin{aligned} \int f d\mu &= \int_C f d\mu + \int_{T \setminus C} f d\mu \leq \lim \int_C f_\alpha d\mu + \mu(T \setminus C) \\ &= \underline{\lim} \left(\int f_\alpha d\mu - \int_{T \setminus C} f_\alpha d\mu \right) + \mu(T \setminus C) \\ &\leq \underline{\lim} \int f_\alpha d\mu + 2\mu(T \setminus C). \end{aligned}$$

In a similar way we get $\int f d\mu \geq \overline{\lim} \int f_\alpha d\mu - 2\mu(T \setminus C)$, from which the result follows. Proof of (c) is straightforward. To prove (d), assume $0 < f < 1$, fix a positive integer n and let $V_i = \{x \in T : f(x) > i/n\}$, $0 \leq i \leq n$. V_i 's are open and

$$\sum_{i=1}^n \frac{i-1}{n} \chi_{V_{i-1} \setminus V_i} \leq f \leq \sum_{i=1}^n \frac{i}{n} \chi_{V_{i-1} \setminus V_i}.$$

Using (c) and the fact

$$\int \left(\sum_{i=1}^n \frac{i}{n} \chi_{V_{i-1} \setminus V_i} - \sum_{i=1}^n \frac{i-1}{n} \chi_{V_{i-1} \setminus V_i} \right) d\mu = \frac{1}{n} \mu(V_0) = \frac{1}{n} \mu(T)$$

we get the result.

COROLLARY 7. *There exists an E -valued τ -smooth measure μ on T which represents φ , if and only if $\mu(P) = 0$, for each Borel set $P \in \mathfrak{B}(X)$, $P \cap T = \emptyset$.*

PROOF. If $\bar{\mu}(P) = 0$ for any $P \in \mathfrak{B}(X)$, $P \cap T = \emptyset$, we define $\mu(Q) = \bar{\mu}(\hat{Q})$ for any $Q \in \mathfrak{B}(T)$, $\hat{Q}$ being in $\mathfrak{B}(X)$ such that $\hat{Q} \cap T = Q$. It is easily verified that μ is well defined and is countably additive. Also, since $\bar{\mu}$ is τ -smooth and $\bar{\mu}(P) = 0$, $\forall P \in \mathfrak{B}(X)$, $P \cap T = \emptyset$, it easily follows that μ is τ -smooth. Further, $\int f d\bar{\mu} = \int f|_X d\mu$, $\forall f \in C(X)$. Conversely, if there is such a μ , then

$$\int f d\bar{\mu} = \int f|_X d\mu, \forall f \in C(X).$$

This means $H = \{f \in B(X): \int f d\bar{\mu} = \int f|_X d\mu\}$ is a monotone order σ -closed subspace of $B(X)$ and contains all bounded upper semicontinuous on X (using Proposition 6). Since the subspace, of $B(X)$, generated by upper semicontinuous functions on X is a vector sublattice of $B(X)$ (simple verification) we have $H = B(X)$ and so the result follows (cf. Corollary 3).

PROPOSITION 8. *A sufficient condition for the existence of a unique τ -smooth E -valued Borel measure μ on T which represents φ is that*

- (i) *whenever $\{f_\alpha\}$ is a decreasing net in $C_b(T)$ with $\inf f_\alpha = 0$ (pointwise order) then $\bigwedge \varphi(f_\alpha) = 0$, and*
- (ii) *E is embedded, as an ordered vector space, in a weakly (σ, ∞) -distributive vector lattice [7], preserving arbitrary suprema and infima.*

PROOF. Idea of proof is same as Proposition 4. The measure $\bar{\mu}: \mathfrak{B}(X) \rightarrow E$ is regular in this case. Proceeding as in Proposition 4 and using Corollary 7, we get the result.

PROPOSITION 9. *A sufficient condition for the existence of a unique τ -smooth E -valued Borel measure μ on T which represents φ is that whenever $\{f_\alpha\}$ is a decreasing net in $C_b(T)$ with $\inf f_\alpha = 0$ (pointwise order), then $\|\varphi(f_\alpha)\| \rightarrow 0$.*

PROOF. As in Proposition 5, $\bar{\phi}(C(X))$ can be considered embedded, as an ordered vector space, in $C(S)$ for a Stonian compact Hausdorff space S , preserving arbitrary suprema and infima. This gives us a positive linear map $\varphi: C_b(T) \rightarrow C(S) \subset B_1(S)$, $B_1(S)$ being all bounded real-valued functions on S with pointwise order. Since $B_1(S)$ is boundedly complete and weakly (σ, ∞) -distributive, using Proposition 8, we get a τ -smooth $B_1(S)$ -valued measure μ_0 on T representing φ . Now $H = \{f \in B(T): \int f d\mu_0 \in B(S)\}$ is a monotone order σ -closed subspace on T and so $H = B(T)$ (same argument as in Corollary 7). The required measure is $\mu = \psi \circ \mu_0|_{\mathfrak{B}(T)}$. To prove it is E -valued let $H_1 = \{f \in B(T): \int f d\mu \in E\}$. Then H_1 is a monotonic order σ -closed subspace of $B(T)$ and contains upper semicontinuous bounded functions on T . Arguing as in Corollary 7, we prove $H_1 = B(T)$. This proves μ is E -valued. Uniqueness is easily verified.

PROPOSITION 10. *A necessary and sufficient condition for φ being representable by a unique tight Borel measure μ on T is that $\varphi(1) = \bar{\mu}(X) = \bigvee \{ \bar{\mu}(C) : C \text{ compact, } C \subset T \}$. If E is a boundedly complete vector lattice then this will happen if and only if for any uniformly bounded net $\{f_\alpha\} \subset C_b(T)$ such that $f_\alpha \rightarrow 0$ uniformly on compact subsets of X_0 , $\varphi(f_\alpha) \rightarrow 0$ in E (order convergence).*

PROOF. First suppose that the condition is satisfied. Let B be any Borel subset of X disjoint from T and let C be any compact subset of T . Then $\bar{\mu}(X) \geq \bar{\mu}(B \cup C) = \bar{\mu}(B) + \bar{\mu}(C)$. Thus $\bar{\mu}(B) = 0$ and so by Corollary 7 there exists a well-defined τ -smooth E -valued Borel measure μ on T with $\bar{\mu}(B) = \mu(B \cap T)$, $\forall B \in \mathfrak{B}(X)$. In particular, $\bar{\mu}(C) = \mu(C)$ for any compact $C \subset T$. If P is a closed subset of X , then for any compact $C \subset T$, we have

$$\mu(P \setminus P \cap C) = \mu(P \setminus C) \leq \mu(T \setminus C) \leq \bar{\mu}(X \setminus C) = \bar{\mu}(X) - \bar{\mu}(C),$$

and so $\mu(P) = \bigvee \{ \mu(P \cap C) : C \text{ compact in } T \}$. Now for any open set U , in T , $\mu(U) = \bigvee \{ \mu(P) : P \subset U, P \text{ closed in } X \}$ (Proposition 6). Hence μ is tight. Converse and uniqueness are easy to verify. Let E be a boundedly complete vector lattice and suppose that φ satisfies the hypothesis. We define a partial order on $I = \{(C, \alpha) : C \text{ a compact subset of } T \text{ and } \alpha \text{ a finite subset of } T \setminus C\}$, $(C_2, \alpha_2) \geq (C_1, \alpha_1)$ if $C_2 \supset C_1$ and $\alpha_2 \supset \alpha_1 \setminus C_2$. I becomes a directed set. Define, $\forall (C, \alpha) \in I$, $f_{C,\alpha} \in C(X)$, $0 \leq f_{C,\alpha} \leq 1$,

$$f_{C,\alpha} = \begin{cases} 0, & \text{on } \alpha, \\ 1, & \text{on } C. \end{cases}$$

Evidently $f_{C,\alpha}|_T \rightarrow 1$ uniformly on compact subsets of T and so $\bar{\mu}(f_{C,\alpha}) \rightarrow \bar{\mu}(1)$ (order convergence) in E . For a $(C_0, \alpha_0) \in I$, $\inf \{ \bar{\mu}(f_{C,\alpha}) : (C, \alpha) \geq (C_0, \alpha_0) \} \leq \inf \{ \bar{\mu}(f_{C_0,\alpha}) : (C_0, \alpha) \geq (C_0, \alpha_0) \} = \bar{\mu}(C_0)$ and so $\varphi(1) = \bar{\mu}(1) = \bigvee \{ \bar{\mu}(C) : C \text{ compact, } C \subset T \}$. The converse is straightforward.

REMARK. The second characterization of Proposition 10 is the definition of the tight functional given in [3].

Using similar methods we have the following sufficient condition for the measure extension to hold in any boundedly σ -complete vector lattice.

PROPOSITION 11. *Let $\mathfrak{A}$ be an algebra of subsets of a set Y , $\mathfrak{A}^\sigma$ be the σ -algebra generated by $\mathfrak{A}$, and E a boundedly monotone σ -complete partially ordered vector space. Let $q : \mathfrak{A} \rightarrow E$ be a positive, finitely additive set function such that whenever $\{A_n\}$ is a monotone decreasing sequence in $\mathfrak{A}$ with $\bigcap_{n=1}^\infty A_n = \emptyset$, then $\|q(A_n)\| \rightarrow 0$. Then there exists a countably additive E -valued measure q^σ on $\mathfrak{A}^\sigma$ which extends q .*

PROOF. As in Proposition 5, we can consider $q(\mathfrak{A}) \subset C(S) \subset B_0(S)$, for some Stonian compact Hausdorff space S . With pointwise order on $B_0(S)$, it is weakly σ -distributive and so we have a countably additive measure $\mu : \mathfrak{A}^\sigma \rightarrow$

$B_0(S)$ [6]. The desired measure is $q^\sigma = \psi_1 \circ \mu$. It is easy to verify that q^σ is E -valued.

I am very grateful to the referee for making many useful suggestions which simplified some proofs.

REFERENCES

1. S. Kaplan, *On the second dual of the space of continuous functions*, Trans. Amer. Math. Soc. **86** (1957), 70–90. MR **19**, 868.
2. S. S. Khurana, *Lattice-valued Borel measures*, Rocky Mountain J. Math. **6** (1976), 377–382.
3. V. S. Varadarajan, *Measures on topological spaces*, Mat. Sb. (N.S.) **55** (97) (1961), 35–100; English transl., Amer. Math. Soc. Transl. (2) **48** (1965), 161–228. MR **26** #6342.
4. J. D. M. Wright, *Stone-algebra-valued measures and integrals*, Proc. London Math. Soc. (3) **19** (1969), 107–122. MR **39** #1625.
5. ———, *Measures with values in a partially ordered vector space*, Proc. London Math. Soc. (3) **25** (1972), 675–688. MR **49** #9152.
6. ———, *The measure extension problem for vector lattices*, Ann. Inst. Fourier (Grenoble) **21** (1971), 65–85. MR **48** #8748.
7. ———, *An algebraic characterization of vector lattices with the Borel regularity property*, J. London Math. Soc. (2) **7** (1973), 277–285. MR **48** #11441.
8. ———, *An extension theorem*, J. London Math. Soc. (2) **7** (1974), 531–539. MR **49** #9153.
9. ———, *Embeddings in vector lattices*, J. London Math. Soc. (2) **8** (1974), 699–706. MR **50** #5439.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242